Topologies on affine schemes.

Recall a morphism $f: T \to S$ is called flat if B is flat as an A-module, where $T = \mathrm{Spec}\, B$ and $S = \mathrm{Spec}\, A$, i.e.

(i) $H^0(B)$ is a flat $H^0(A)$-module. &

(ii) $\forall i \in \mathbb{Z}$,

$$\cancel{H^i(B)} \quad H^i(A) \underset{H^0(A)}{\otimes} H^0(B) \xrightarrow{\sim} H^i(B).$$

(ii)' $\forall i \in \mathbb{Z}$, $\forall M \in \mathrm{Mod}\, A$

$$H^i(A) \underset{H^0(A)}{\otimes} H^0(M) \xrightarrow{\sim} H^i(M).$$

(ii)'' $N \in \mathrm{Mod}_A^{\heartsuit}$, $\Rightarrow$ $B \underset{A}{\otimes} N \in \mathrm{Mod}_B^{\heartsuit}$.

Notice if $S' \to S$ is flat and $S \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$, then.

$$H^i(\mathcal{O}_{S'}) \simeq H^i(\mathcal{O}_S) \underset{H^0(\mathcal{O}_S)}{\otimes} H^0(\mathcal{O}_{S'}) \Rightarrow S' \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}.$$

Exercise: $f: S' \to S$ is flat $\Leftrightarrow$ $\forall n \geq 0$ $\tau^{\leq n}(f): {}^{\leq n}S \to {}^{\leq n}S'$ is flat.

Def'n: A morphism $f: S' \to S$ of affine schemes is:

- flat of finite presentation (ffp) (ppf). if: f is flat & ${}^{cl}f: {}^{cl}S \to {}^{cl}S'$ is of finite presentation;
- smooth if: f is flat & ${}^{cl}f$ is smooth;
- étale if: f is flat & ${}^{cl}f$ is étale; (i.e. ${}^{cl}f$ smooth & $\Omega_{H^0(B)/H^0(A)} = 0$).
- open embedding if: f is flat & ${}^{cl}f$ is an open embedding;
- Zariski if: f is flat & ${}^{cl}f$ is a disjoint union of open embeddings.

The following says that the étale site does not depend on the derived structure of an affine scheme.

Prop: $\mathrm{Sch}^{\mathrm{aff}}_{/S} \to \mathrm{Sch}^{\mathrm{aff}}_{/{}^{cl}S}$ gives an equivalence between

$$(S' \to S) \mapsto S' \underset{S}{\times} {}^{cl}S \to {}^{cl}S$$

the subcategories of $S' \xrightarrow{f} S$ étale & $S'_0 \xrightarrow{f_0} {}^{cl}S$ étale.
Moreover, f is an open embedding (Z.) $\iff$ f_0 is an open embedding (Z.).

<u>Idea of argument</u>: (i) For each $n \geq 0$, established

$$Sch^{aff}_{\text{étale in } \leq^{(n+1)}S} \to Sch^{aff}_{\text{étale in } \leq^n S}$$

$$(S'_{n+1} \to {}^{\leq(n+1)}S) \mapsto S'_{n+1} \underset{\leq(n+1)S}{\times} {}^{\leq n}S \quad \text{is an equivalence.}$$

(ii) deformation theory describe $\{ S'_{n+1} \in {}^{\leq(n+1)}Sch^{aff} \mid \tau^{\leq n}(S'_{n+1}) \simeq S'_n \}$ is equivalent to $\{$sq.-zero extensions of S'_n by $\mathcal{I} \in QCoh(S'_n)^{\heartsuit}[n+1]\}$ i.e. Hom$(T^* S'_n, \mathcal{I}[1])$.
then étale has consequences to $T^* S'_n$.

<u>Def'n</u>: A morphism $f: S' \to S$ is a flat / ppf (smooth, étale, Zariski) <u>covering</u> if f is —— & ${}^{cl}f: {}^{cl}S' \to {}^{cl}S$ is surjective.

These properties have the usual behavior with respect to base change and are local when expected to be so.
E.g.: - pullback of flat is flat ; - $f: S' \to S$ a map & $S'' \to S$
sm. ét. sm. a cover, then $f'': S' \underset{S}{\times} S'' \to S''$ flat/sm./ét.
(but not Zariski) $\Rightarrow$ f is the same.
- $g: S' \to S$ being flat, ppf, sm., ét, Z. is local w.r.t. flat topology on S'.

<u>Def'n</u>: A map $f: \mathcal{X}_1 \to \mathcal{X}_2$ of prestacks is <u>affine schematic</u> if for every $S \to \mathcal{X}_2$, $S \in Sch^{aff}$, $S \underset{\mathcal{X}_2}{\times} \mathcal{X}_1$ is an affine scheme.

- f is <u>flat</u> (<u>ppf</u>, <u>smooth</u>, <u>étale</u>, <u>open embedding</u>, <u>Zariski</u>) if f is affine schematic & for every S the induced map
$$\mathcal{X}_1 \underset{\mathcal{X}_2}{\times} S \to S \quad \text{is so.}$$

<u>Descent condition & stacks.</u>

Notation: given a map $S' \to S$ the Čech nerve is the simplicial object.
$$(S'/S)^{\bullet}: \Delta^{op} \to Sch^{aff}$$

pictured as:

$$\cdots \; \substack{\to\\\to\\\to\\\to} \; S' \times_S S' \times_S S' \; \substack{\to\\\to\\\to} \; S' \times_S S' \; \substack{\to\\\to} \; S' .$$

$$2 \qquad\qquad 1 \qquad\qquad 0$$

<u>Def'n:</u> A prestack $\mathcal{X}$ <u>satisfy</u> flat (ppf, étale, sm, etc.) <u>descent</u> if for every flat covering $S' \to S$, the canonical morphism

$$\mathcal{X}(S) \xrightarrow{\simeq} \lim_{\Delta^{op}} \left(\mathcal{X}(S') \rightrightarrows \mathcal{X}(S' \times_S S') \; \substack{\to\\\to\\\to} \cdots \right).$$

$$\| $$

$$\mathrm{Tot}(\mathcal{X}(S'^{\bullet}/S)).$$

We let Stk denote the category of prestack that <u>satisfy étale</u> descent.

<u>Rk:</u> One could choose other topologies, but there is a sense in which this is the most general choice.

~~Rk~~ <u>Lemma:</u> The natural inclusion $\mathrm{Stk} \underset{\perp}{\overset{L}{\rightleftarrows}} \mathrm{PStk}$ has a left adjoint, i.e. sheafification. Moreover, L is left exact, i.e. commutes w/ finite limits.

<u>Idea:</u> L is obtained by inverting <u>étale equivalences</u>, i.e. $f: \mathcal{X}_1 \to \mathcal{X}_2$ in PStk s.t.

$$\mathrm{Hom}_{\mathrm{PStk}}(\mathcal{X}_2, \mathcal{Y}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{PStk}}(\mathcal{X}_1, \mathcal{Y}). \quad \forall\, \mathcal{Y} \in \mathrm{Stk}.$$

<u>Notation:</u> [GR-I] have the practice of considering all objects as prestacks, even stacks. So they denote by L the composite $\mathrm{PStk} \xrightarrow[L]{} \mathrm{Stk} \hookrightarrow \mathrm{PStk}$. We might do the same.

<u>Example:</u> The functor $\mathrm{QCoh}^*(-) : (\mathrm{Sch}^{\mathrm{aff}})^{op} \to \mathrm{Cat}_\infty^{\mathrm{s.t.}}$ satisfies flat descent, i.e.

For $A \to A'$ flat,

$$\mathrm{Mod}_A \xrightarrow{\simeq} \lim_{\Delta^{op}} \left(\mathrm{Mod}_{A'} \rightrightarrows \mathrm{Mod}_{A' \otimes_A A'} \; \substack{\to\\\to\\\to} \cdots \right).$$

<u>Idea:</u> (i) reduce the statement to $\mathrm{Mod}_A^{\leq 0}$, use Barr-Beck-Lurie Theorem. and prove its conditions by hand.

(ii) - $\lim_{n \in \Delta^{op}} \mathrm{Mod}_{A'^{\otimes_A (n+1)}}$ has a t-structure.

- $\mathrm{Mod}_A \rightleftarrows \lim_{\Delta^{op}} \mathrm{Mod}_{(A'/A)^\bullet}$ is t-exact.

- reduce to a statement about the heart of the t-structure, which is the usual statement.

<u>Proposition</u> (Sanity check): $\mathrm{Sch}^{aff} \hookrightarrow \mathrm{PStk}$ factors through Stk

<u>Idea:</u> let $T \in \mathrm{Sch}^{aff}$ & $S' \to S$ be an étale covering.
we have a map:

$$T(S) \to T(S') \rightrightarrows T(S' \times_S S') \equiv \cdots$$

$$\parallel$$

$$(\Delta) \quad \mathrm{Maps}(S,T) \to \mathrm{Maps}(S',T) \rightrightarrows \mathrm{Maps}(S' \times_S S', T) \equiv \cdots$$

let $T = \mathrm{Spec}\, B$, $S = \mathrm{Spec}\, A$, $S' = \mathrm{Spec}\, A'$, (Δ) corresponds to:

$$\mathrm{Hom}_{\mathrm{CAlg}}(B,A) \to \mathrm{Hom}_{\mathrm{CAlg}}(B,A') \rightrightarrows \mathrm{Hom}_{\mathrm{CAlg}}(B, A' \otimes_A A') \equiv \cdots$$

$$\mathrm{Tot}(\mathrm{Hom}_{\mathrm{CAlg}}(B, A'/A^\bullet)) \simeq \mathrm{Hom}_{\mathrm{CAlg}}(B, \mathrm{Tot}(A'/A^\bullet)) . \simeq \mathrm{Hom}_{\mathrm{CAlg}}(B,A). \text{ as required.}$$

Now $\mathrm{CAlg} \xrightarrow[\mathrm{oblv}]{} \mathrm{Vect}$ preserves limits.

Since $A \to A'$ is also flat, flat descent $\Rightarrow$ $\mathrm{Tot}(A'/A)^\bullet \simeq A$. ∎

<u>RK</u>: The operation $L : \mathrm{PStk} \to \mathrm{Stk} \hookrightarrow \mathrm{PStk}$ sometimes is hard to control. There is one concept which helps with understanding it, we might not need it, but I will include it for completeness.

<u>Def'n:</u> A morphism $f: \mathcal{X}_1 \to \mathcal{X}_2$ in PStk is an étale surjection if $\forall_{x_2 : S \to \mathcal{X}_2}$, $\exists \phi : S' \to S$ étale s.t.

$$\phi^*(x_2) : S' \to \mathcal{X}_2 \quad \text{belongs to the essential image of}$$

$$f(S') : \mathcal{X}_1(S') \to \mathcal{X}_2(S').$$

Here is an example of how this notion is used.

<u>Prop:</u> Let $\mathcal{X}_1 \to \mathcal{X}_2$ be an étale surjection then

$$|\mathcal{X}_1/\mathcal{X}_2|_{PStk} \to \mathcal{X}_2 \quad \text{is an étale equivalence, i.e.}$$

$$|L(\mathcal{X}_1/\mathcal{X}_2)|_{Stk} = \underbrace{|L(\mathcal{X}_1)/L(\mathcal{X}_2)|_{Stk} = L(|\mathcal{X}_1/\mathcal{X}_2|_{PStk})}_{\text{by defn.}} \to L(\mathcal{X}_2)$$

is an equivalence.

<u>Cor:</u> $f: \mathcal{X}_1 \to \mathcal{X}_2$ in PStk is affine schematic (flat, ppf, smooth, étale, open embedding) then so is $L(f): L(\mathcal{X}_1) \to L(\mathcal{X}_2)$.

<u>Conditions on stacks.</u>

<u>Q:</u> How does descent relate to the conditions of n-coconnective convergent, laft, lft & k-truncated?

We will discuss the n-coconnective condition. and leave the others for the interested reader to look into it.

Let $^{\leq n}Stk := \left\{ \mathcal{X} \in {}^{\leq n}PStk := Fun(({}^{\leq n}Sch^{aff})^{op}, Spc) \;\middle|\; \mathcal{X} \text{ satisfy étale descent w.r.t. the étale top. on } {}^{\leq n}Sch^{aff} \right\}$.

$^{\leq n}L : {}^{\leq n}PStk \to {}^{\leq n}Stk \to {}^{\leq n}PStk$ The sheafification.

<u>Rk:</u> Assume $\mathcal{X} \in {}^{\leq n}PStk$ is k-truncated, then.

$$^{\leq n}L\mathcal{X} \simeq \mathcal{X}^{+(k+2)}, \text{ where.}$$

$$\mathcal{X}^+(S) := \operatorname*{colim}_{\substack{S' \to S \text{ étale} \\ \text{cover.}}} \operatorname{Tot}(\mathcal{X}(S'/S)).$$

In particular, $^{\leq n}L\mathcal{X}$ is k-truncated. if $\mathcal{X}$ is k-truncated.

Rk 2: If $\mathcal{X}$ is convergent, it might be tricky to say if $L\mathcal{X}$ is convergent.
However, if $\mathcal{X} \in Stk$ then ${}^{conv}\mathcal{X}$, (i.e. the convergent prestack associated to $\mathcal{X}$) is a stack.

Notice: $LKE_{{}^{\leq n}Sch^{aff} \to Sch^{aff}} : {}^{\leq n}PStk \to PStk$ does not send ${}^{\leq n}Stk$ to Stk. (trying to commute totalizations w/ arbitrary colimit.)

2 Def'n: A stack $\mathcal{X} \in Stk$ is <u>n-coconnective</u> if the map:

$$\underline{\underline{L^{\circ}}}LKE_{{}^{\leq n}Sch^{aff} \hookrightarrow Sch^{aff}}({}^{\leq n}\mathcal{X}) \to \mathcal{X} \quad \text{is an equivalence.}$$

$$\| $$

$${}^{L}LKE_{{}^{\leq n}Sch^{aff} \hookrightarrow Sch^{aff}}.$$

In particular, a classical stack, i.e. 0-coconnective is not recovered from ${}^{cl}Sch^{aff}$ simply by LKE via ${}^{cl}Sch^{aff} \hookrightarrow Sch^{aff}$, one also has to sheatify in <u>derived</u> aff. schemes the result.

But n-coconnective prestack + stack $\not\Rightarrow$ n-coconnective stack.
E.g. $\mathbb{P}^1 \in {}^{cl}Stk$ is an étale sheaf, unclear $LKE_{{}^{cl}Sch^{aff} \hookrightarrow Sch^{aff}}(\mathbb{P}^1)$ satisfy descent.
Similar subtleties happen for the finite type condition. See § 2.7 in Chapter 2 [GR-I].

Example: Let $T = Spec\, B \in Sch^{aff}$, we know $T \in Stk$ by Prop. above.

Q: When is T a 0-coconnective ^{pre}stack? $\quad S = Spec\, A$, $S_0 = Spec\, A_0$

Consider
$$LKE_{{}^{\leq 0}Sch^{aff} \hookrightarrow Sch^{aff}}({}^{cl}T)(S) = \underset{S \to S_0}{colim}\, \underset{Maps(S_0, {}^{cl}T)}{\overset{\|}{{}^{cl}T(S_0)}} = \underset{\substack{A_0 \to A \\ A_0 \in CAlg^{disc.}}}{colim} Hom_{CAlg^{disc.}}(\tau^{\geq 0}(B), A_0)$$

$$\simeq \underset{A_0 \to A}{colim}\, Hom_{CAlg}(B, A_0).$$

Now, any $A \in CAlg$ is a sifted colimit of discrete algebras (from its presentation as SCR)

So if B is a compact & projective object of CAlg, then

Spec B is 0-coconnective.

FACT: $B \in$ CAlg is compact & projective iff B is a retract of $\mathrm{Sym}(M)$ for $M \simeq k^{\oplus n}$ for some $n \geq 0$, where

$$\mathrm{CAlg} \underset{\mathrm{oblv}}{\overset{\mathrm{Sym}}{\rightleftarrows}} \mathrm{Vect}$$ is the left adjoint to forgetful functor.

Idea: General fact about (presentable) ∞-categories:

$G: \mathcal{C} \to \mathcal{D}$ if G preserves limits, sifted colimits & is conservative.

Then (i) $\exists (F, G)$
(ii) F takes compact proj. objects to compact proj. objects.
(iii) $\{D_\alpha\}$ a set of gen. then $\{F(D_\alpha)\}$ a set of cpt. proj. gen.
(iv) $\mathcal{D}$ proj. gen. $\Rightarrow$ $\mathcal{C}$ proj. gen.

FACT 2. All of the above hold for oblv.

In our example, let $\mathrm{CAlg}_0 :=$ cat. gen. by $\underset{k[x]}{\mathrm{Sym}(k)}$ under colimits.

$\mathrm{CAlg}_0 \hookrightarrow \mathrm{CAlg}$ has a right adjoint U.

Claim: U is an equivalence, i.e. $U(A) \to A$ is an isom.
enough to check $\mathrm{oblv}(U(A)) \to \mathrm{oblv}(A)$ is an isom.

Or
$$\begin{array}{ccc}\mathrm{Hom}_{\mathrm{Vect}}(k, \mathrm{oblv}_0(U(A))) & = & \mathrm{Hom}_{\mathrm{CAlg}}(\mathrm{Sym}(k), U(A)) \\ \downarrow & & \wr\wr \\ \mathrm{Hom}_{\mathrm{Vect}}(k, \mathrm{oblv}_0(A)) & \leftarrow & \mathrm{Hom}_{\mathrm{CAlg}_0}(\mathrm{Sym}(k), A).\end{array}$$

is an eq. but $\mathrm{oblv}_0(U(A)) = \mathrm{oblv}(A)$. So this is clear!